

# Pricing of Contingent Claims Under the Real-World Measure

Shane Michael MILLER

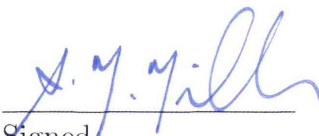
A thesis submitted for the degree of  
Doctor of Philosophy  
at the University of Technology, Sydney.  
30 September 2007.

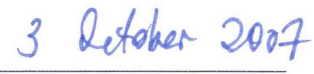
# Certificate of Authorship

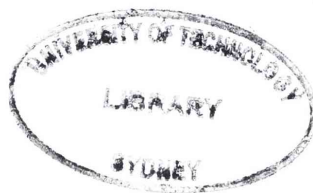
I certify that the work of this thesis has not previously been submitted for a degree nor has it been submitted as part of the requirements for a degree except as fully acknowledged within the text.

I also certify that the thesis has been written by me. Any help that I have received in my research work and the preparation of the thesis itself has been acknowledged. In addition I certify that all information sources and literature used are indicated in the thesis.

The work of Chapter 6 is based on the research article of Miller & Platen (2005) while three key results that appear in Hulley, Miller & Platen (2005) were based on work included in Chapters 2, 3 and 5. Hulley, Miller & Platen (2007) further extends computations within the previous paper. It is also expected that as a minimum, Chapters 4 and 5 will also form the basis of further journal articles.

  
Signed

  
Date



# Contents

Certificate	i
Acknowledgements	i
Table of Contents	ii
List of Figures	vi
Basic Notation	ix
Abstract	xiii
<b>1 Introduction</b>	<b>1</b>
<b>2 Real-World Pricing and the Benchmark Approach</b>	<b>7</b>
2.1 Continuous Financial Market . . . . .	8
2.2 Growth Optimal Portfolio . . . . .	12
2.3 Real-World Pricing . . . . .	17
2.4 Forward Prices and Contract Valuation . . . . .	26
2.5 European Option Pricing . . . . .	31
<b>3 A Black-Scholes-Merton Model</b>	<b>38</b>
3.1 Black-Scholes-Merton Dynamics . . . . .	39
3.2 Zero-Coupon Bond Prices . . . . .	42
3.3 Options on the GOP . . . . .	44
3.4 Options on an Exchange Price . . . . .	47

3.5	Interest Rate Options . . . . .	54
<b>4</b>	<b>A Modified Constant Elasticity of Variance Model</b>	<b>57</b>
4.1	Modified CEV Dynamics . . . . .	58
4.2	Zero-Coupon Bond Prices . . . . .	62
4.3	Options on the GOP . . . . .	68
4.4	Options on an Exchange Price . . . . .	77
4.5	Interest Rate Options . . . . .	85
<b>5</b>	<b>A Stylised Minimal Market Model</b>	<b>92</b>
5.1	Stylised MMM Dynamics . . . . .	93
5.2	Zero-Coupon Bond Prices . . . . .	101
5.3	Options on the GOP . . . . .	105
5.4	Options on an Exchange Price . . . . .	110
5.5	Interest Rate Options . . . . .	115
<b>6</b>	<b>An Interest Rate Term Structure Model</b>	<b>124</b>
6.1	Stylised Continuous Financial Market . . . . .	125
6.2	Interest Rate Term Structure Model . . . . .	130
6.3	Interest Rate Options . . . . .	138
<b>7</b>	<b>An Extended Minimal Market Model</b>	<b>146</b>
7.1	Extended MMM Dynamics . . . . .	147
7.2	Zero-Coupon Bond Prices . . . . .	149
7.3	Options on the GOP . . . . .	154
7.4	Options on an Exchange Price . . . . .	158
7.5	Interest Rate Options . . . . .	164
<b>8</b>	<b>Conclusion</b>	<b>170</b>
<b>A</b>	<b>Selected Results on Stochastic Processes</b>	<b>172</b>

B Selected Probability Distributions	174
C Contingent Claim Prices: Summary Tables	182
Bibliography	186

# List of Figures

2.1	Logarithm of the WSAI in USD from 1925 to 2003. . . . .	14
2.2	Candidate Radon-Nikodym derivative in USD from 1925 to 2003. . . . .	22
3.1	WSAI volatility $ \theta_t^i $ in USD from 1926 to 2003 under the BSMM. . . . .	40
3.2	Transition density for $\bar{T} \in [0, 1]$ year under the BSMM. . . . .	41
3.3	‘Total’ volatility $\sigma_{ij}$ with inputs $\sigma_i = 0.15$ , $\sigma_j \in [0.05, 0.25]$ and $\rho \in [-1, +1]$ under the BSMM. . . . .	53
4.1	Transition density for $\bar{T} \in [0, 1]$ year with $\beta_i \in \{0.25, 0.5, 0.75\}$ under the MCEV model. . . . .	62
4.2	$M_{\bar{T}}^i(0)$ with $\beta \in [0, 1)$ for $\bar{T} \in [0, 30]$ (LHS) and $\bar{T} \in [0, 100]$ (RHS) under the MCEV model. . . . .	65
4.3	Upper and lower bounds for $P^i(0, \bar{T})$ for $\bar{T} \in [0, 30]$ (LHS) and $\bar{T} \in [0, 100]$ (RHS) under the MCEV model. . . . .	65
4.4	$m_{\bar{T}}^i(0)$ for $\beta_i \in [0, 1)$ and $\bar{T} \in [0, 30]$ under the MCEV model. . . . .	66
4.5	$m_{\bar{T}}^i(0)$ for $ \theta_0^i  \in [0.05, 0.50]$ and $\bar{T} \in [0, 30]$ under the MCEV model with $\beta_i \in \{0.0, 0.25, 0.5, 0.75\}$ , clockwise. . . . .	68
4.6	Implied volatility surfaces for options on the GOP under the MCEV model with $\bar{T} \in [0, 30]$ for $\beta_i \in \{0.0, 0.25, 0.5, 0.75\}$ , clockwise. . . . .	75
4.7	Implied volatility surfaces for exchange price options under the MCEV model with $\bar{T} \in [0, 30]$ for $\beta \in \{0.0, 0.25, 0.5, 0.75\}$ , clockwise. . . . .	84
4.8	ATM interest rate caplet implied volatility term structure under the MCEV with $\bar{T} \in [0, 30]$ for $\beta \in \{0.0, 0.25, 0.5, 0.75\}$ , clockwise. . . . .	90
4.9	WSAI volatility $ \theta_t^i $ in USD from 1926 to 2003 for two CEV exponent scenarios $\beta_i \in \{0.50, 1.15\}$ under the MCEV model. . . . .	91
5.1	Empirical $\langle \sqrt{\tilde{S}^{i, \delta_*}} \rangle_t$ in USD from 1929 to 2003 and theoretical fit under the SMMM. . . . .	95

5.2	WSAI volatility $ \theta_t^i $ in USD from 1925 to 2003 under the SMMM.	97
5.3	Difference between transitions densities of SMMM and BSMM for $\bar{T} \in [0, 1]$ (LHS) and $\bar{T} \in [0, 30]$ (RHS) years.	98
5.4	$p_{\chi^2}(x; 0, \lambda)$ and absorption probability, for $\lambda = 2, 5, 10, 25$ clockwise.	100
5.5	$M_{\bar{T}}^i(0)$ for $\bar{T} \in [0, 30]$ (LHS) and $\bar{T} \in [0, 100]$ (RHS) under the SMMM.	103
5.6	$m_{\bar{T}}^i(0)$ for $ \theta_0^i  \in [0.05, 0.50]$ and $\bar{T} \in [0, 30]$ under the SMMM.	105
5.7	Implied volatility surfaces for options on the GOP under the SMMM to $\bar{T} \in \{10, 30, 50, 100\}$ years, clockwise.	110
5.8	Implied volatility surfaces for exchange price options under the SMMM to $\bar{T} \in \{10, 30, 50, 100\}$ years, clockwise.	116
5.9	ATM interest rate caplet implied volatility term structure under the SMMM to $\bar{T} \in \{10, 30, 50, 100\}$ years, clockwise.	123
6.1	Logarithm of WSAI in JPY from 1949 to 2003.	126
6.2	Standardised covariation $\rho_{\ln(\bar{S}^{i,\delta_*}), r^i}(t)$ for JPY from 1960 to 2003.	127
6.3	Candidate Radon-Nikodym derivative in JPY from 1949 to 2003.	128
6.4	Empirical $\langle \sqrt{\bar{S}^{i,\delta_*}} \rangle_t$ in JPY from 1949 to 2003 and theoretical fit under the SMMM.	129
6.5	Market and hypothetical shadow rates for Japan from 1995 to 2003.	132
6.6	$P^i(0, \bar{T})$ and $G_{\bar{T}}^i(0, r_0^i)$ in JPY as at 1 July 2004.	134
6.7	$f^i(0, \bar{T})$ and $g_{\bar{T}}^i(0, r_0^i)$ scenarios under the SMMM with stochastic short rate.	137
6.8	ATM implied cap volatilities for the JPY market, together with Gaussian and CIR++ implementations.	142
6.9	Implied cap volatility surface for the JPY market, under the SMMM with truncated Gaussian short rate to $\bar{T} \in [0, 10]$ years.	143
6.10	Differences in real-world and risk-neutral implied cap volatility surface, under the SMMM with truncated Gaussian short rate.	144
7.1	$M_{\bar{T}}^i(0)$ with $\nu_i \in [3, 6]$ for $\bar{T} \in [0, 30]$ (LHS) and $\bar{T} \in [0, 100]$ (RHS) under the EMMM.	151
7.2	$m_{\bar{T}}^i(0)$ for $\nu_i \in [3, 6]$ and $\bar{T} \in [0, 30]$ under the EMMM.	153

7.3	$m_{\bar{T}}^i(0)$ for $ \theta_t^i  \in [0.05, 0.50]$ and $\bar{T} \in [0, 30]$ with $\nu_i \in \{3, 4, 5, 6\}$ clockwise, under the EMMM. . . . .	154
7.4	Implied volatility surfaces for options on the GOP under the EMMM with $\bar{T} \in [0, 30]$ years for $\nu_i \in \{3, 4, 5, 6\}$ , clockwise. . . . .	159
7.5	Implied volatility surfaces for exchange price options under the EMMM with $\bar{T} \in [0, 30]$ years for $\nu \in \{3, 4, 5, 6\}$ , clockwise. . . . .	165



# Notation and Abbreviations

## Mathematical Notation

$(a, b)^+$	maximum of $a$ and $b$ ;
$\mathbf{x}^\top$	transpose of a vector or matrix $\mathbf{x}$ ;
$ \mathbf{x} $	absolute value of $\mathbf{x}$ or Euclidean norm;
$\mathbf{A} = [a^{i,j}]_{i,j=1}^{k,d}$	$(k \times d)$ -matrix $\mathbf{A}$ with $i, j$ th component $a^{i,j}$ ;
$\mathbf{A}^{-1}$	inverse of a matrix $\mathbf{A}$ ;
$\infty$	infinity;
$(a, b)$	open interval $a < x < b$ in $\mathbb{R}$ ;
$[a, b]$	closed interval $a \leq x \leq b$ in $\mathbb{R}$ ;
$\mathbb{R} = (-\infty, +\infty)$	set of real numbers;
$\mathbb{R}^+ = [0, +\infty)$	set of non-negative real numbers;
$\mathbb{R}^d$	$d$ -dimensional Euclidean space;
$\Omega$	sample space;
$A \cup B$	the union of sets $A$ and $B$ ;
$\langle X, Y \rangle_t$	covariation of processes $X$ and $Y$ at time $t$ ;
$\langle X \rangle_t$	quadratic variation of process $X$ at time $t$ ;
$n! = 1 \cdot 2 \cdot \dots \cdot n$	factorial of $n$ ;
$\mathbb{I}_{\{A\}}$	indicator function for every event $A$ to be true;
$\mathcal{A}$	collection of events, sigma-algebra;
$\mathcal{A}$	filtration;
$E[X]$	expectation of $X$ ;
$E[X   \mathcal{A}]$	conditional expectation of $X$ ;
$P[A]$	probability of $A$ ;
$P[A   B]$	probability of $A$ conditioned on $B$ ;
$\in$	element of;
$\neq$	not equal to;
$\mathbf{1}$	unit matrix;
$\ln(a)$	natural logarithm of $a$ ;
$\lim_{N \rightarrow \infty}$	limit as $N$ tends to infinity.

## Statistical Notation

$\Gamma(\cdot)$	gamma function;
$\Gamma(\cdot; \cdot)$	incomplete gamma function;
$\mathcal{Q}(\cdot; \cdot)$	regularised incomplete gamma function;
$\mathcal{Q}^{-1}(\cdot; \cdot)$	inverse of the regularised incomplete gamma function;
$\mathcal{N}(\cdot)$	standard Gaussian distribution function;
$p_{\mathcal{N}}(\cdot)$	standard Gaussian density;
$\mathcal{G}(\cdot; \cdot)$	two-parameter gamma complementary distribution function;
$p_{\mathcal{G}}(\cdot; \cdot)$	two-parameter gamma density;
$\chi^2(\cdot; \cdot)$	central chi-square distribution function;
$p_{\chi^2}(\cdot; \cdot)$	central chi-square density;
$\chi^2(\cdot; \cdot, \cdot)$	non-central central chi-square distribution function;
$p_{\chi^2}(\cdot; \cdot, \cdot)$	non-central chi-square density;
$\mathcal{I}(\cdot; \cdot, \cdot; \cdot, \cdot)$	doubly non-central Beta distribution function;
$p_{\mathcal{I}}(\cdot; \cdot, \cdot; \cdot, \cdot)$	doubly non-central Beta density;
$q(t, z_t; T, z_T)$	transition density to move from $z_t$ at time $t$ to $z_T$ at time $T > t$ ; used in this text for the GOP;
$p(t, x_t; T, x_T)$	transition density to move from $x_t$ at time $t$ to $x_T$ at time $T > t$ ; used in this text for the discounted GOP;
$I_{\nu}(\cdot)$	modified Bessel function of the first kind with index $\nu$ .

## Abbreviations

GOP	growth optimal portfolio;
BSMM	Black-Scholes-Merton model;
MCEV	modified constant elasticity of variance;
SMMM	stylised minimal market model;
EMMM	extended minimal market model;
IRTS	interest rate term structure;
WSAI	world stock accumulation index;
SDE	stochastic differential equation;
OTM	out-of-the-money;
ATM	at-the-money;
ITM	in-the-money;
LHS	left-hand-side;
RHS	right-hand-side.

## Thesis Specific Notation

$S_t^{i,j}$	primary security account for $j$ th asset in the $i$ th currency at time $t$ ;
$B_t^i$	savings account in the $i$ th currency at time $t$ ;
$r_t^i$	short rate of the $i$ th currency at time $t$ ;
$\psi_t^i$	shadow short rate of the $i$ th currency at time $t$ ;
$ \theta_t^i $	market price of risk in the $i$ th currency at time $t$ ; or volatility of the GOP in the $i$ th currency at time $t$ ;
$X_t^{i,j}$	exchange price of the $j$ th asset in units of the $i$ th currency at time $t$ ;
$S_t^{i,\delta}$	non-negative wealth process (portfolio) in the $i$ th currency at time $t$ ;
$S_t^{i,\delta*}$	Growth Optimal Portfolio (GOP) in the $i$ th currency at time $t$ ;
$\bar{S}_t^{i,\delta*}$	discounted GOP in the $i$ th currency at time $t$ ;
$\hat{S}_t^\delta$	benchmarked portfolio at time $t$ ;
$\Lambda_t^{i,\theta}$	candidate Radon-Nikodym derivative in the $i$ th currency at time $t$ ;
$P_{i,\theta}$	candidate risk-neutral probability measure in the $i$ th currency;
$H_T^i$	contingent claim in the $i$ th currency at time $T$ ;
$U_t^{i,H_T^i}$	derivative price of contingent claim $H_T^i$ in the $i$ th currency at time $t$ ;
$P^i(t, T)$	zero-coupon bond price with maturity $T$ in the $i$ th currency at time $t$ ;
$M_T^i(t)$	discounted GOP contribution to the zero-coupon bond price with maturity $T$ in the $i$ th currency at time $t$ ; or, expected value of the candidate Radon-Nikodym derivative in the $i$ th currency at time $t$ ;
$G_T^i(t)$	short rate contribution to the zero-coupon bond price with maturity $T$ in the $i$ th currency at time $t$ ;
$f^i(t, T)$	forward rate with maturity $T$ in the $i$ th currency at time $t$ ;
$m_T^i(t)$	discounted GOP contribution to the forward rate with maturity $T$ in the $i$ th currency at time $t$ ;
$g_T^i(t)$	short rate contribution to the forward rate with maturity $T$ in the $i$ th currency at time $t$ ;
$U_t^i$	price process in the $i$ th currency at time $t$ ;
$V_{T,K,U^i}^i(t)$	derivative price of the process $U^i$ with expiry $T$ and strike price $K$ in the $i$ th currency at time $t$ ;
$F_{T,U^i}^i(t)$	forward price of $U^i$ with maturity $T$ in the $i$ th currency at time $t$ ;

## Thesis Specific Notation (continued)

$A_{T,K,U^i}^{i,\pm}(t)$	European asset binary option on the process $U^i$ with expiry $T$ and strike price $K$ in the $i$ th currency at time $t$ ;
$B_{T,K,U^i}^{i,\pm}(t)$	European bond binary option on the process $U^i$ with expiry $T$ and strike price $K$ in the $i$ th currency at time $t$ ;
$c_{T,K,U^i}^i(t)$	European call option on the process $U^i$ with expiry $T$ and strike price $K$ in the $i$ th currency at time $t$ ;
$p_{T,K,U^i}^i(t)$	European put option on the process $U^i$ with expiry $T$ and strike price $K$ in the $i$ th currency at time $t$ ;
$\mathbf{zbc}_{\bar{T},T,K}^i(t)$	European call option on a zero-coupon bond with expiry $\bar{T}$ , maturity $T \geq \bar{T}$ and strike price $K$ in the $i$ th currency at time $t$ ;
$\mathbf{zbp}_{\bar{T},T,K}^i(t)$	European put option on a zero-coupon bond with expiry $\bar{T}$ , maturity $T \geq \bar{T}$ and strike price $K$ in the $i$ th currency at time $t$ ;
$\mathbf{cap}_{\mathcal{T},K,N}^i(t)$	interest rate cap with the set of dates $\mathcal{T} = \{T_0, T_1, \dots, T_n\}$ , strike rate $K$ and notional $N$ in the $i$ th currency at time $t \leq T_0$ ;
$\mathbf{cpl}_{T_{\ell-1},T_{\ell},K}^i(t)$	$\ell$ th individual interest rate caplet within $\mathbf{cap}_{\mathcal{T},K,N}^i(t)$ ;
$\mathbf{flr}_{\mathcal{T},K,N}^i(t)$	interest rate floor with the set of dates $\mathcal{T} = \{T_0, T_1, \dots, T_n\}$ , strike rate $K$ and notional $N$ in the $i$ th currency at time $t \leq T_0$ ;
$\mathbf{fl}_{T_{\ell-1},T_{\ell},K}^i(t)$	$\ell$ th individual interest rate floorlet within $\mathbf{flr}_{\mathcal{T},K,N}^i(t)$ .

# Abstract

The aim of this thesis is to price contingent claims under the real-world probability measure. Real-world pricing results naturally by selecting the numeraire as the growth optimal portfolio (GOP). Under this approach, the existence of an equivalent risk-neutral probability measure is not required. Furthermore, the GOP can be used to define other basic contingent claims, such as exchange prices, primary security accounts, and even zero-coupon bonds. We begin with application of the real-world pricing formula to derive forward prices for each of these financial quantities. The obtained formulae are model independent, yet reveal important differences between the real-world and classical risk-neutral approaches.

Real-world prices are systematically derived under each of the models studied within this thesis for the following contingent claims: zero-coupon bonds; options on the GOP; options on exchange prices; and interest rate caps and floors via options on zero-coupon bonds. We start with the classic Black-Scholes-Merton model, where the GOP follows a geometric Brownian motion. Under this model, real-world pricing recovers the results of classical risk-neutral pricing, since the corresponding Radon-Nikodym derivative is a martingale.

For each of the remaining models studied, the GOP is based on a time-transformed squared Bessel process. In each case, real-world prices may differ from classical risk-neutral prices because the candidate Radon-Nikodym derivative is a strict supermartingale. The second model considered proposes a modified form of the constant elasticity of variance model for the GOP. New analytic results for zero-coupon bonds and options on the GOP are derived that were previously analysed using numerical methods. Real-world prices for options on exchange prices and interest rate derivatives are also provided.

Three versions of the minimal market model are also examined. This model class overcomes some of the deficiencies of the aforementioned approaches since the dynamics for the GOP better reflect empirical market features, such as leptokurtic returns, the leverage effect and a stochastic yet stationary volatility structure. Under a stylised version of the minimal market model with a constant short rate, we derive analytic solutions to the complete suite of contingent claims examined within the thesis. We subsequently allow the short rate to be stochastic in order to accurately model the term structure of interest rates, with a focus on low interest rate environments. The proposed model provides a very good fit to interest rate



cap data from the Japanese market. Finally, we examine an extended version of the minimal market model, which contains one additional constant parameter and reverts to a constant short rate. Further analytic prices are derived under the real-world measure for zero-coupon bonds, options on the GOP and options on exchange prices while semi-analytic prices are available for interest rate caps and floors. The results for the extended minimal market model generalise those of the stylised model. They also possess a similar structural form to those derived for the modified constant elasticity of variance model. Hence the extended minimal market model summarises all of the features of models where the GOP is based on time-transformed squared Bessel processes.